# The Error Bound of the Perturbation of the Drazin Inverse

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ABSTRACT

Let $A$ and $E$ be $n \times n$ matrices and $B = A + E$. Denote the Drazin inverse of $A$ by $A^D$. In this paper we give an upper bound for the relative error $\|B^D - A^D\|/\|A^D\|_2$ and a lower bound for $\|B^D\|_2$ under certain circumstances. The continuity properties and the derivative of the Drazin inverse are also considered.

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## I. INTRODUCTION

A necessary and sufficient condition for the continuity of the Drazin inverse (to be defined in detail in Section II) was proved by Campbell and Meyer in 1974 [1], but no explicit bound has yet been found. In Campbell's 1974 paper, he stated the main result: Suppose that $A_j$, $j = 1, 2, \dots,$ and $A$ are $n \times n$ complex matrices such that $A_j \to A$. Then $A_j^D \to A^D$ (where $A^D$ is the Drazin inverse of $A$) if and only if there is a real number $j_0$ such that $\operatorname{corerank} A_j = \operatorname{corerank} A$ for $j \geqslant j_0$ [where $\operatorname{corerank} A = \operatorname{rank} A^{i(A)}$ and the index $i(A)$ of $A$ is defined as the smallest integer $k \geqslant 0$ such that the range of $A^k$ equals the range of $A^{k+1}$, i.e. $R(A^k) = R(A^{k+1})$].

In the same paper, Campbell indicated two difficulties in establishing norm estimates for the Drazin inverse: First, the Drazin inverse has a weaker type of "cancellation law" and is somewhat harder to work with algebraically than the Moore-Penrose. Also complicating things is the fact that the Jordan form is not a continuous function from $\mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ and the Drazin inverse can be thought of in terms of the Jordan canonical form.

Again in [2] (1979), Campbell gave some idea about the condition number with respect to the Drazin inverse, expressed in terms of Jordan form: If one is going to calculate $A^D$ by $A^D = A^k(A^{2k+1})^+A^k$, where $A^+$ is the Moore-




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Penrose inverse of $A$, then rather than $\|A\|(\|A^D\|+1)$ or some such, a much better idea of the conditioning would be

$$C(A)=\|P\|\,\|P^{-1}\|(\|J\|^k+\|J^+\|^k),$$

where $PJP^{-1}$ is the Jordan form of $A$.

In this paper we shall give an explicit bound for $\|B^D-A^D\|/\|A^D\|_2$ in terms of $A$, $A^D$, and $E=B-A$, provided $E$ is sufficiently small when rank $B^k=$ rank $A^k$, where $k=\max\,(i(A),i(B))$. On the other hand, if rank $B^k$ $>$ rank $A^k$, we shall find a lower bound for $\|B^D\|_2$ which tends to infinity as $B$ approaches $A$.

In the next section some mathematical background that will be needed later is introduced.

## II. PRELIMINARY

It is assumed that readers are familiar with the concepts and results listed below.

(a) For every matrix $A\in\mathbb{C}^{n\times n}$, there exists a unique matrix $A^D\in\mathbb{C}^{n\times n}$, satisfying

$$A^{h+1}A^D=A^h \qquad \text{for some positive integer } h, \tag{1}$$

$$A^DAA^D=A^D, \tag{2}$$

$$AA^D=A^DA. \tag{3}$$

The matrix $A^D$ is called the Drazin inverse of $A$.

(b) For every matrix $A\in\mathbb{C}^{m\times n}$, there exists a unique matrix $A^+\in\mathbb{C}^{n\times m}$, satisfying

$$\begin{aligned} A^+AA^+&=A,\\ AA^+A&=A^+,\\ (AA^+)^H&=AA^+,\\ (A^+A)^H&=A^+A, \end{aligned} \tag{4}$$

where $A^H$ is the conjugate transpose of $A$. The matrix $A^+$ is called the

Moore-Penrose inverse of $A$.

(c)

(i) Let rank $A = \operatorname{rank}(A+E), \|A^+\|_2 \|E\|_2 < 1$. Then

$$\|(A+E)^+\|_2 \leqslant \frac{\|A^+\|_2}{1-\|A^+\|_2 \|E\|_2} \tag{5}$$

where $\|\cdot\|_2$ denotes the spectral norm of $A$, i.e.,

$$\|A\|_2 = \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1}} \|Ax\|_2.$$

(ii) Let $\operatorname{rank}(A+E) > \operatorname{rank} A$. Then

$$\|(A+E)^+\|_2 \geqslant \frac{1}{\|E\|_2}. \tag{6}$$

(d) Let $k = i(A)$. For every positive integer $h \geqslant k$, (a) holds and

$$A^D = A^h (A^{2h+1})^+ A^h. \tag{7}$$

(e) Unless it is indicated specifically, a (fixed) unitarily invariant norm is applied throughout this paper. Thus, for every matrix $A \in \mathbb{C}^{n\times n}$ and every unitary matrix $U \in \mathbb{C}^{n\times n}$, $\|UAU^H\| = \|A\|$.

Further, the following two inequalities are always used: for every matrix $A \in \mathbb{C}^{n\times n}$,

$$\begin{gathered} \|A\|_2 \leqslant \|A\|, \\ \|AB\| \leqslant \|A\|_2 \|B\| \text{ or } \|A\| \, \|B\|_2. \end{gathered} \tag{8}$$

For the details of Moore-Penrose inverse and Drazin inverse, see the books by Ben-Israel and Greville [3] and by Campbell and Meyer [2]. The proof of (c) has appeared in Noble's paper [4]. Mirsky has given a systematic treatment of unitarily invariant norms [5].

## III. THE CASE rank $A^k = \text{rank}\,(A+E)^k$

Let $A, E \in \mathbf{C}^{n\times n}$, $B = A + E$, and for any arbitrary positive integer $h$, define $E(A^h)$ by $B^h - A^h = E(A^h)$. Then $\|B^h\|_2 \leqslant \|A^h\|_2 + \mathcal{E}(A^h)$, where

$$\mathcal{E}(A^h) = \sum_{i=0}^{h-1} C_h^i \|A\|_2^i \|E\|_2^{h-i} \geqslant \|E(A^h)\|_2$$

and $C_h^i$ is the binomial coefficient.

LEMMA 1. *Let* rank $B^h =$ rank $A^h$ *and* $\|(A^h)^+\|_2 \mathcal{E}(A^h) < 1$. *Then*

$$\|(B^h)^+\|_2 \leqslant \frac{\|(A^h)^+\|_2}{1 - \|(A^h)^+\|_2 \mathcal{E}(A^h)}.$$

*Proof.* $\|(A^h)^+\|_2 \|E(A^h)\|_2 \leqslant \|(A^h)^+\|_2 \mathcal{E}(A^h) < 1$. Thus by (5), we have

$$\|(B^h)^+\|_2 = \|[A^h + E(A^h)]^+\|_2 \leqslant \frac{\|(A^h)^+\|_2}{1 - \|(A^h)^+\|_2 \|E(A^h)\|_2}$$
$$\leqslant \frac{\|(A^h)^+\|_2}{1 - \|(A^h)^+\|_2 \mathcal{E}(A^h)}. \qquad \blacksquare$$

THEOREM 1. *Let* $k = i(B)$, rank $A^k =$ rank $B^k$, *and* $\|(A^{2k+1})^+\|_2 \mathcal{E}(A^{2k+1}) < 1$. *Then*

$$\|B^D\|_2 \leqslant \frac{\|(A^{2k+1})^+\|_2 [\|A^k\|_2 + \mathcal{E}(A^k)]^2}{1 - \|(A^{2k+1})^+\|_2 \mathcal{E}(A^{2k+1})}.$$

*Proof.* Since $B^D = B^k (B^{2k+1})^+ B^k$,

$$\|B^D\|_2 \leqslant \|B^k\|_2^2 \|(B^{2k+1})^+\|_2$$
$$\leqslant \frac{\|(A^{2k+1})^+\|_2}{1 - \|(A^{2k+1})^+\|_2 \mathcal{E}(A^{2k+1})} [\|A^k\|_2 + \mathcal{E}(A^k)]^2, \qquad \text{by Lemma 1.}$$

$\blacksquare$

Theorem 1 tells us $B^D=(A+E)^D$ is bounded in the neighborhood of $\|E\|$ provided rank $A^{i(B)}=$ rank $B^{i(B)}$. This is one of the bases for deriving the error estimate for the Drazin inverse, and the second is contained in the asymptotic expansion of $B^D-A^D$. It can be derived as follows:

Let $k=\max\{i(A),i(B)\}$. Then

$$\begin{aligned}
B^D-A^D &= -B^DEA^D+\left[B^D-A^D+B^D(B-A)A^D\right]\\
&= -B^DEA^D+B^D(1-AA^D)-(1-B^DB)A^D\\
&= -B^DEA^D+(B^D)^{k+1}B^k(1-AA^D)-(1-B^DB)A^k(A^D)^{k+1}\\
&= -B^DEA^D+(B^D)^{k+1}\left[A^k+E(A^k)\right](1-AA^D)\\
&\quad -(1-B^DB)\left[B^k-E(A^k)\right](A^D)^{k+1}\\
&= -B^DEA^D+(B^D)^{k+1}E(A^k)(1-AA^D)\\
&\quad +(1-B^DB)E(A^k)(A^D)^{k+1}
\end{aligned} \tag{9}$$

By taking $\|\cdot\|$ of both sides of (9) and noticing (8),

$$\begin{aligned}
\|B^D-A^D\| &\leqslant \|B^D\|_2\|A^D\|_2\|E\|+\|B^D\|_2^{k+1}\left(1+\|AA^D\|_2\right)\|E(A^k)\|\\
&\quad +\left(1+\|B^DB\|_2\right)\|A^D\|_2^{k+1}\|E(A^k)\|,
\end{aligned}$$

where

$$\|E(A^k)\|\leqslant\sum_{i=0}^{k-1}C_k^i\|A\|_2^i\|E\|^{k-i}.$$

Let $\mathcal{E}(A^{2k+1})\|(A^{2k+1})^+\|_2<1$ and rank $A^k=$ rank $B^k$. Then by Theorem 1,

$$B^D=A^D+O(\|E\|).$$

Substitute $B^D$ in the right hand side of (9):

$$\begin{aligned}
B^D - A^D &= -\left[A^D + O(\|E\|)\right]EA^D + \left[A^D + O(\|E\|)\right]^{k+1}E(A^k)(1-AA^D)\\
&\quad + \left\{1 - \left[A^D + O(\|E\|)\right](A+E)\right\}E(A^k)(A^D)^{k+1}\\
&= -A^DEA^D + (A^D)^{k+1}\sum_{i=0}^{k-1} A^iEA^{k-1-i}(1-AA^D)\\
&\quad + (1-A^DA)\sum_{i=0}^{k-1} A^iEA^{k-1-i}(A^D)^{k+1} + O(\|E\|^2)\\
&= -A^DEA^D + \sum_{i=0}^{k-1} (A^D)^{i+2}EA^i(1-AA^D)\\
&\quad + (1-AA^D)\sum_{i=0}^{k-1} A^iE(A^D)^{i+2} + O(\|E\|^2). \qquad \blacksquare \qquad (10)
\end{aligned}$$

We are now in a position to prove the theorem bounding $\|B^D - A^D\|/\|A^D\|_2$.

THEOREM 2. *Let* $k = \max\{i(A), i(B)\}$, rank $A^k$ = rank $B^k$, *and* $\|(A^{2k+1})^+\|_2\mathfrak{S}(A^{2k+1}) < 1$. *Then*

$$\frac{\|B^D - A^D\|}{\|A^D\|_2} \leqslant C(A)\frac{\|E\|}{\|A\|_2} + o(\|E\|),$$

*where the condition number*

$$C(A) = \left[2\sum_{i=0}^{k-1} \|(A^D)^{i+1}\|_2\|A^i\|_2\left(1 + \|A\|_2\|A^D\|_2\right) + \|A^D\|_2\right]\|A\|_2,$$

*and* $o(\|E\|)$ *is of the second order in* $E$.

*Proof.* Take $\|\cdot\|$ of both sides of (10) to get that

$$\|B^D - A^D\| \leqslant \left[2\sum_{i=0}^{k-1} \|(A^D)^{i+2}\|_2\|A^i\|_2\left(1 + \|A\|_2\|A^D\|_2\right) + \|A^D\|_2^2\right]\|E\|$$
$$+ \|O(\|E\|^2)\|,$$

and the result is obvious. $\blacksquare$

Theorem 2 additionally suggests that $k$ may have an effect on $\|B^D - A^D\|/\|A^D\|_2$, if one notes that $C(A) \geqslant 4k+1$, where the 4 results from the "crude" estimate $\|A\|\,\|A^D\| \geqslant \|AA^D\| \geqslant 1$.

Moreover, by Theorem 2, we get the sufficient condition for continuity.

COROLLARY 2. *Let* $\|E_i\| \to 0$, $k_i = \max\{i(A), i(A+E_i)\}$. *If there exists a positive integer* $i_o$ *such that for every* $i \geqslant i_o$, $\operatorname{rank} A^{k_i} = \operatorname{rank}(A+E_i)^{k_i}$, *then* $(A+E_i)^D \to A^D$.

The derivative of $A^D$ can also be derived from (10). Define corerank $A = \operatorname{rank}(A^{i(A)})$, $k = i(A)$. If $A(t)$ is a differentiable matrix function and corerank $A(t) =$ constant, $i(A(t)) \leqslant k$, for every $t$, then

$$\frac{dA^D}{dt} = \sum_{i=0}^{k-1}\left[(A^D)^{i+2}\frac{dA}{dt}A^i(1-A^DA)+(1-AA^D)A^i\frac{dA}{dt}(A^D)^{i+2}\right]$$
$$-A^D\frac{dA}{dt}A^D.$$

These two results have been obtained by Campbell and Meyer by different methods [1, 6].

## IV. THE CASE rank $A^k < \operatorname{rank}(A+E)^k$

The next lemma, which we have used to obtain a vector $y \in R(A+E)^k \cap N(A^k)$, where $N(A)$ is the null space of $A$, is proved by Campbell and Meyer [1].

LEMMA 2. *Let* $S$, $T$ *be subspaces* $\subseteq \mathbb{C}^n$, $\dim S > \dim T$, *and* $\mathbb{C}^n = T \oplus T'$. *Then there exists* $z \neq 0$, $z \in S \cap T'$.

THEOREM 3. *Let* $B = A+E$, $k = \max\{i(A), i(B)\}$, $\operatorname{rank} B^k > \operatorname{rank} A^k$. *Then*

$$\|B^D\|_2 \geqslant \frac{1}{\mathfrak{S}(A^k)^{1/k}}.$$

*Proof.* Note that $R(A^k) \oplus N(A^k) = \mathbb{C}^n$. Thus, by Lemma 2, $\operatorname{rank} B^k > \operatorname{rank} A^k$ implies there exists $y \neq 0$, $y \in R(B^k) \cap N(A^k)$. Without loss of generality, we can assume $\|y\|_2 = 1$.

Again note that $B^D B$ is a projection on $R(B^k)$ along $N(B^k)$ and $B^D B = (B^D)^k B^k$. Then

$$\begin{aligned} 1 &= y^H y = y^H (B^D)^k B^k y \\ &= y^H (B^D)^k E(A^k) y \\ &\leqslant \| B^D \|_2^k \mathcal{E}(A^k), \qquad \text{by the Cauchy inequality.} \end{aligned}$$

In other words,

$$\| B^D \|_2 \geqslant \frac{1}{\mathcal{E}(A^k)^{1/k}}. \qquad \blacksquare$$

The following example shows this lower bound is sharp for $k = 2$. It is related to an example in [2, p. 232].

Example. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$A^D = 0 \text{ and } (A + E)^D = \begin{bmatrix} 1/\varepsilon & (1/\varepsilon)^2 \\ 0 & 0 \end{bmatrix}$$

and $i(A) = 2$, $i(A + E) = 1$. Thus $k = 2$ and $\operatorname{rank}(A + E)^2 = 1 > \operatorname{rank} A^2 = 0$. By noting that $\| A \|_2^2$ is the maximum eigenvalue of $A^T A$, we have $\|(A + E)^D \|_2 = (1/\varepsilon)\sqrt{1 + (1/\varepsilon)^2}$, $\mathcal{E}(A^2) = \| E \|_2^2 + 2\| A \|_2 \| E \|_2 = \varepsilon^2 + 2\varepsilon$. Hence

$$\lim_{\varepsilon \to \infty} \|(A + E)^D \|_2 \mathcal{E}(A^2)^{1/2} = 1.$$

Proposition. *Let* $\| E_i \|_2 \to 0$, $k_i = \max\{i(A + E_i), i(A)\}$. *Then there exists a positive integer* $i_o$ *such that for every* $i \geqslant i_o$, $\operatorname{rank}(A + E_i)^{k_i} \geqslant \operatorname{rank} A^{k_i}$.

*Proof.* Assume there exists an infinite subsequence $\{A + E_{i_j}\}$ such that

$$\operatorname{rank}(A + E_{i_j})^{k_{i_j}} < \operatorname{rank} A^{k_{i_j}},$$

or alternatively,

$$\operatorname{rank}(A + E_{i_j})^{n} < \operatorname{rank} A^{n}.$$

Apply (6), and we have

$$\|(A^n)^+\|_2 \geq \frac{1}{\mathcal{E}_{i_j}(A^n)} \to \infty \qquad \text{when} \quad j \to \infty,$$

a contradiction. ■

By this proposition and Theorem 3, we can prove the necessary condition for the continuity that was proved in Campbell and Meyer [1], in a different way.

Corollary 4. *Let* $\|E_i\|_2 \to 0$, $k_i = \max\{i(A), i(A + E_i)\}$, *and* $\operatorname{rank}(A + E_i)^{k_i} \neq \operatorname{rank} A^{k_i}$. *Then* $(A + E_i)^D \nrightarrow A^D$.

*Proof.* By the last proposition, $\operatorname{rank}(A + E_i)^{k_i} \neq \operatorname{rank} A^{k_i}$ implies that there exists a positive integer $i_o$ such that for every $i \geq i_o$, $\operatorname{rank}(A + E_i)^{k_i} > \operatorname{rank} A^{k_i}$. By Theorem 3, $\|(A + E_i)^D\|_2 \geq 1/\mathcal{E}(A^{k_i})^{1/k_i} \to \infty$.

## V. CLOSING REMARK

The decomposition $A^D = A^k(A^{2k+1})^+A^k$ shows that the continuity for $(A^{2k+1})^+$ is essential for the continuity for $A^D$. So in our paper we have considered the perturbation $\mathcal{E}(A^{2k+1})$ of $A^{2k+1}$ as a whole instead of the perturbation $\|E\|$ of $A$. The derivation of the norm estimate for the Drazin inverse turns out surprisingly simple and conventional, and we have worked out the continuity theorem and the differentiation of $A^D$ from a unified point of view.

## REFERENCES

1 S. L. Campbell and C. D. Meyer, Continuity properties of the Drazin inverse, *Linear Algebra Appl.* 10:77–83 (1975).

2 S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*, Pitman, 1979.
3 A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley, 1974.
4 B. Noble, Method for computing the Moore-Penrose generalized inverse and related matters, in *Generalized Inverse and its Application*, (M. Z. Nashed, Ed.), 1976.
5 L. Mirsky, Symmetric gauge functions and unitarily invariant norms, *Quart. J. Math. Oxford Ser.* (2) 11, No. 2 (1960).
6 S. L. Campbell, Differentiation of the Drazin inverse, *Linear and Multilinear Algebra* 8, No. 3 (1980).
7 G. W. Stewart, On the perturbation of pseudo-inverses, projections and linear least squares problems, *SIAM Rev.* 19, No. 4 (Oct. 1977).

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